Improving Coupling of Multiscale Computational Models within the Adaptive Hydraulics Framework

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Overview

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Introduction

Adaptive Hydraulics (AdH):
- 2D and 3D shallow water (SW) equations
- 3D Navier-Stokes equations
- Barotropic/baroclinic transport of constituents (salinity, temperature, sediments, etc.)
- Spatiotemporal adaptivity

(US Army Corps of Engineers)
Objective

“To algebraically couple 2D and 3D shallow water models in AdH, in a conservative manner”
Solution method

Spatial discretization of PDE’s to generate nonlinear ODE’s in time
- Streamline upwind Petrov-Galerkin (SUPG) method

Temporal discretization of nonlinear ODE’s to generate a system of nonlinear equations
- Up to second order implicit finite difference method

Nonlinear equations solved using Newton-Raphson iterative method
AdH: 2D SW models

Equations after applying SUPG

- Depth integrated continuity equation (1)

\[
\begin{align*}
\sum_e \left[ \int_{\Omega^2_e} \left( \frac{\partial h}{\partial t} - \nabla^2 \phi_i \cdot (\nabla h) \right) d\Omega^2_e + \int_{\partial \Omega^2_e} \left[ \phi_i (\nabla h) \cdot \mathbf{n} \right] d\partial \Omega^2_e + \frac{\tilde{p}_c}{\tilde{e}} \right] &= 0
\end{align*}
\]

- Depth integrated horizontal momentum equations (2) and (3)

\[
\begin{align*}
\sum_e \left[ \int_{\Omega^2_e} \left( \frac{\partial (\bar{u}h)}{\partial t} - \frac{\partial \phi_i}{\partial x} \left( \bar{u}\bar{u}h - \frac{gh^2}{2} - \frac{h\sigma_{xx}}{\rho} \right) - \frac{\partial \phi_i}{\partial y} \left( \bar{v}\bar{v}h - \frac{h\sigma_{xy}}{\rho} \right) \right) d\Omega^2_e + \int_{\partial \Omega^2_e} \left[ \phi_i n_x \left( \bar{u}\bar{u}h - \frac{gh^2}{2} - \frac{h\sigma_{xx}}{\rho} \right) + \phi_i n_y \left( \bar{v}\bar{v}h - \frac{h\sigma_{xy}}{\rho} \right) \right] d\partial \Omega^2_e + \tilde{p}_c^{mx} \right] &= 0
\end{align*}
\]

\[
\begin{align*}
\sum_e \left[ \int_{\Omega^2_e} \left( \frac{\partial (\bar{v}h)}{\partial t} - \frac{\partial \phi_i}{\partial x} \left( \bar{v}\bar{v}h - \frac{gh^2}{2} - \frac{h\sigma_{yy}}{\rho} \right) - \frac{\partial \phi_i}{\partial y} \left( \bar{v}\bar{v}h - \frac{h\sigma_{yy}}{\rho} \right) \right) d\Omega^2_e + \int_{\partial \Omega^2_e} \left[ \phi_i n_x \left( \bar{v}\bar{v}h - \frac{gh^2}{2} - \frac{h\sigma_{yy}}{\rho} \right) + \phi_i n_y \left( \bar{v}\bar{v}h - \frac{h\sigma_{yy}}{\rho} \right) \right] d\partial \Omega^2_e + \tilde{p}_c^{my} \right] &= 0
\end{align*}
\]
AdH: 3D SW models

Equations after applying SUPG

- Depth summed continuity equation (4)

\[
\sum_{i \in C} \sum_{e} \left[ \int_{\Omega^3_e} \left( \nabla \phi_i \cdot \mathbf{v} \right) d\Omega^3_e \right] + \sum_{e} \left[ \int_{\partial \Omega^{2D}_{e,s}} \left[ \phi_i \frac{\partial \eta}{\partial t} n_x \right] d\Omega^{2D}_{e,s} + \int_{\partial \Omega^{2D}_{e,b}} \left[ \phi_i \frac{\partial b}{\partial t} n_x \right] d\Omega^{2D}_{e,b} \right] + \sum_{i \in C} \sum_{e} \int_{\partial \Omega^{2D}_{e,v}} [\phi_i \mathbf{n} \cdot \mathbf{v}] d\Omega^{2D}_{e,v} + \sum_{i \in C} \sum_{e} \frac{P^c_e}{\rho} = 0
\]

- Horizontal momentum equations (5) and (6)

\[
\sum_{e} \left[ \frac{\partial}{\partial t} \int_{\Omega^3_e} \phi_i \left( v \cdot \mathbf{n} \right) d\Omega^3_e \right] + \int_{\Omega^3_e} \left[ -\nabla \phi_i \cdot \mathbf{v} - \phi_i f + \frac{\partial \phi_i}{\partial x} \frac{P}{\rho_0} + \nabla \phi_i \cdot \mathbf{\tau} \right] d\Omega^3_e + \int_{\partial \Omega^{2D}_{e,v}} [\phi_i \mathbf{n} \cdot \mathbf{v}] d\Omega^{2D}_{e,v} = 0
\]

\[
\sum_{e} \left[ \frac{\partial}{\partial t} \int_{\Omega^3_e} \phi_i \left( v \cdot \mathbf{n} \right) d\Omega^3_e \right] + \int_{\Omega^3_e} \left[ -\nabla \phi_i \cdot \mathbf{v} + \phi_i f + \frac{\partial \phi_i}{\partial y} \frac{P}{\rho_0} + \nabla \phi_i \cdot \mathbf{\tau} \right] d\Omega^3_e + \int_{\partial \Omega^{2D}_{e,v}} [\phi_i \mathbf{n} \cdot \mathbf{v}] d\Omega^{2D}_{e,v} = 0
\]
AdH: Final equations

2D SW models:

\[
\begin{align*}
    r_{1,2D}^1 &= \left( r_{1,2D}^{mx}(s_{2D}(t^{n+1})) = 0 \\
    r_{1,2D}^m &= \left( r_{1,2D}^{my}(s_{2D}(t^{n+1})) = 0 \\
    r_{1,2D}^c &= \left( r_{1,2D}^c(s_{2D}(t^{n+1})) = 0
\end{align*}
\]

\[
\begin{align*}
    r_{2,2D}^1 &= \left( r_{2,2D}^{mx}(s_{2D}(t^{n+1})) = 0 \\
    r_{2,2D}^m &= \left( r_{2,2D}^{my}(s_{2D}(t^{n+1})) = 0 \\
    r_{2,2D}^c &= \left( r_{2,2D}^c(s_{2D}(t^{n+1})) = 0
\end{align*}
\]

\[
\vdots
\]

\[
\begin{align*}
    r_{N,2D}^c &= \left( r_{N,2D}^c(s_{2D}(t^{n+1})) = 0
\end{align*}
\]

where

\[
s_{2D} = \{\bar{u}_1, \bar{v}_1, h_1, ..., \bar{u}_N, \bar{v}_N, h_N\}^T
\]

3D SW models:

\[
\begin{align*}
    r_{1,3D}^1 &= \left( r_{1,3D}^{mx}(s_{3D}(t^{n+1})) = 0 \\
    r_{1,3D}^m &= \left( r_{1,3D}^{my}(s_{3D}(t^{n+1})) = 0 \\
    r_{1,3D}^c &= \left( r_{1,3D}^c(s_{3D}(t^{n+1})) = 0
\end{align*}
\]

\[
\begin{align*}
    r_{2,3D}^1 &= \left( r_{2,3D}^{mx}(s_{3D}(t^{n+1})) = 0 \\
    r_{2,3D}^m &= \left( r_{2,3D}^{my}(s_{3D}(t^{n+1})) = 0 \\
    r_{2,3D}^c &= \left( r_{2,3D}^c(s_{3D}(t^{n+1})) = 0
\end{align*}
\]

\[
\vdots
\]

\[
\begin{align*}
    r_{N,3D}^c &= \left( r_{N,3D}^c(s_{3D}(t^{n+1})) = 0
\end{align*}
\]

where

\[
s_{3D} = \{u_1, v_1, d_1, ..., u_N, v_N, d_N\}^T
\]
AdH: Final equations

Nonlinear equations in vector form for both 2D and 3D models can be written as (8)

\[ R(s_{n+1}) = 0 \]  

(8)

Newton-Raphson iterations (9) are set up to solve (8)

\[ \left( \frac{\partial R}{\partial s_{n+1}} \right)^{(i)} \Delta s_{n+1}^{(i+1)} = -R(s_{n+1}^{(i)}) \]  

(9)

\[ s_{n+1}^{(i+1)} = s_{n+1}^{(i)} + \Delta s_{n+1}^{(i+1)} \]
Newton-Raphson iterations (9) for 2D SW models, for example, look like (10)

\[
\begin{bmatrix}
\frac{\partial r_1^{mx}}{\partial u_1} & \frac{\partial r_1^{mx}}{\partial v_1} & \frac{\partial r_1^{mx}}{\partial h_1} & \frac{\partial r_1^{mx}}{\partial u_2} & \frac{\partial r_1^{mx}}{\partial v_2} & \frac{\partial r_1^{mx}}{\partial h_2} & \ldots & \frac{\partial r_1^{mx}}{\partial h_N} \\
\frac{\partial r_1^{my}}{\partial u_1} & \frac{\partial r_1^{my}}{\partial v_1} & \frac{\partial r_1^{my}}{\partial h_1} & \frac{\partial r_1^{my}}{\partial u_2} & \frac{\partial r_1^{my}}{\partial v_2} & \frac{\partial r_1^{my}}{\partial h_2} & \ldots & \frac{\partial r_1^{my}}{\partial h_N} \\
\frac{\partial r_1^{c}}{\partial u_1} & \frac{\partial r_1^{c}}{\partial v_1} & \frac{\partial r_1^{c}}{\partial h_1} & \frac{\partial r_1^{c}}{\partial u_2} & \frac{\partial r_1^{c}}{\partial v_2} & \frac{\partial r_1^{c}}{\partial h_2} & \ldots & \frac{\partial r_1^{c}}{\partial h_N} \\
\frac{\partial r_2^{mx}}{\partial u_1} & \frac{\partial r_2^{mx}}{\partial v_1} & \frac{\partial r_2^{mx}}{\partial h_1} & \frac{\partial r_2^{mx}}{\partial u_2} & \frac{\partial r_2^{mx}}{\partial v_2} & \frac{\partial r_2^{mx}}{\partial h_2} & \ldots & \frac{\partial r_2^{mx}}{\partial h_N} \\
\frac{\partial r_2^{my}}{\partial u_1} & \frac{\partial r_2^{my}}{\partial v_1} & \frac{\partial r_2^{my}}{\partial h_1} & \frac{\partial r_2^{my}}{\partial u_2} & \frac{\partial r_2^{my}}{\partial v_2} & \frac{\partial r_2^{my}}{\partial h_2} & \ldots & \frac{\partial r_2^{my}}{\partial h_N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial r_N^{c}}{\partial u_1} & \frac{\partial r_N^{c}}{\partial v_1} & \frac{\partial r_N^{c}}{\partial h_1} & \frac{\partial r_N^{c}}{\partial u_2} & \frac{\partial r_N^{c}}{\partial v_2} & \frac{\partial r_N^{c}}{\partial h_2} & \ldots & \frac{\partial r_N^{c}}{\partial h_N} \\
\end{bmatrix}^{(i)} \{ \Delta \bar{u}_1^{(i+1)} \Delta \bar{v}_1 \Delta \bar{v}_2 \ldots \Delta \bar{h}_N \} = - \begin{bmatrix}
\bar{r}_1^{mx} (s_{n+1}^{(i)}) \\
\bar{r}_1^{my} (s_{n+1}^{(i)}) \\
\bar{r}_1^{c} (s_{n+1}^{(i)}) \\
\bar{r}_2^{mx} (s_{n+1}^{(i)}) \\
\bar{r}_2^{my} (s_{n+1}^{(i)}) \\
\bar{r}_2^{c} (s_{n+1}^{(i)}) \\
\vdots \\
\bar{r}_N^{c} (s_{n+1}^{(i)})
\end{bmatrix}
\]
## Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{N}^{2D}$</td>
<td>Set of all nodes in the 2D model</td>
</tr>
<tr>
<td>$\mathcal{J}^{2D}$</td>
<td>Set of all nodes in the 2D model that lie on the 2D-3D interface</td>
</tr>
<tr>
<td>$\mathbf{R}_{{2D}}$</td>
<td>Global residual vector of the 2D domain</td>
</tr>
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<td>$\mathbf{r}<em>{{2D}}^i = {r^{{mx}}</em>{{i,2D}}, r^{{my}}<em>{{i,2D}}, r^c</em>{{i,2D}}}^T$</td>
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</tr>
<tr>
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</tr>
</tbody>
</table>
Notation

\[ R = \{ R^{N_{2D}-g_{2D}}, R^{N_{3D}-g_{3D}}, R^{g_{2D}}, R^{g_{3D}} \}^T \]

(Rearranged) global residual vector of the coupled domain

\[ s = \{ s^{N_{2D}-g_{2D}}, s^{N_{3D}-g_{3D}}, s^{g_{2D}}, s^{g_{3D}} \}^T \]

(Rearranged) global solution vector of the coupled domain

\[ r_i = \{ r_i^{m_x}, r_i^{m_y}, r_i^c \}^T \]

New residual vector at node \( i \) of the coupled domain

\[ s^i = \begin{cases} s_{2D}^i, & \text{if } i \in N_{2D} \\ s_{3D}^i, & \text{if } i \in N_{3D} \end{cases} \]

Solution vector at node \( i \) of the coupled domain

\[ c_i \]

Linear constraint applied at node \( i \) of the 3D interface

\[ C = \{ c_i \}^T \]

Constraint vector for all nodes on the 3D interface

\[ \mathcal{K} \]

A node on the 2D model interface

\[ C(\mathcal{K}) \]

The column (set) of 3D model interface nodes that \( \mathcal{K} \) is coupled to
Example

Primary set definitions:
\[ \mathcal{N}^{2D} = \{ \text{All nodes in 2D model} \} \]
\[ \mathcal{N}^{3D} = \{ \text{All nodes in 3D model} \} \]

Separate Interface Nodes:
\[ \mathcal{J}^{2D} = \{1_{2D}, 2_{2D}, 3_{2D} \} \]
\[ \mathcal{J}^{3D} = \{1_{3D}, 2_{3D}, 3_{3D}, 4, ..., 9 \} \]

Node Columns:
\[ \mathcal{C}(\mathcal{K} = 1_{2D}) = \{1_{3D}, 2_{3D}, 3_{3D} \} \]
\[ \mathcal{C}(\mathcal{K} = 2_{2D}) = \{4, 5, 6 \} \]
\[ \mathcal{C}(\mathcal{K} = 3_{2D}) = \{7, 8, 9 \} \]
Coupling: Old system

Newton iterations (9) for the combined, reordered (non-coupled) 2D-3D system are given by (11)

\[
\begin{bmatrix}
\frac{\partial R_{2D}^{N^2D-j^2D}}{\partial s_{2D}^{N^2D-j^2D}} & [0] & \frac{\partial R_{2D}^{N^2D-j^2D}}{\partial s_{2D}^{j^2D}} & [0] \\
[0] & \frac{\partial R_{3D}^{N^3D-j^3D}}{\partial s_{3D}^{N^3D-j^3D}} & \frac{\partial R_{3D}^{N^3D-j^3D}}{\partial s_{3D}^{j^3D}} & [0] \\
\frac{\partial R_{2D}^{j^2D}}{\partial s_{2D}^{N^2D-j^2D}} & [0] & \frac{\partial R_{2D}^{j^2D}}{\partial s_{2D}^{j^2D}} & [0] \\
[0] & \frac{\partial R_{3D}^{j^3D}}{\partial s_{3D}^{N^3D-j^3D}} & \frac{\partial R_{3D}^{j^3D}}{\partial s_{3D}^{j^3D}} & [0]
\end{bmatrix}
\begin{bmatrix}
\Delta s_{2D}^{N^2D-j^2D} \\
\Delta s_{3D}^{N^3D-j^3D} \\
\Delta s_{2D}^{j^2D} \\
\Delta s_{3D}^{j^3D}
\end{bmatrix}
= -
\begin{bmatrix}
R_{2D}^{N^2D-j^2D}(s_{2D}) \\
R_{3D}^{N^3D-j^3D}(s_{3D}) \\
R_{2D}^{j^2D}(s_{2D}) \\
R_{3D}^{j^3D}(s_{3D})
\end{bmatrix}
\]  

(11)
Coupling: New residuals

Define the ‘new,’ coupled residuals using (12)

\[
R \ni r^j = \begin{cases} 
  r^j_{2D} & \forall j \in N^{2D} - J^{2D} \\
  r^j_{3D} & \forall j \in N^{3D} - J^{3D} \\
  r^j_{2D} + \sum_{i \in C(j)} r^i_{3D} & \forall j \in J^{2D} \\
  s^j_{3D} - s^j_{2D} & \forall j \in J^{3D}, \text{where } \exists! K \in J^{2D} : j \in C(K)
\end{cases}
\]

In vector form, (12) can be informally rewritten as (13)

(Note: \( \Sigma^* \) is not a function, just notation)

\[
\begin{align*}
R_{N^{2D} - J^{2D}} &= R_{2D}^{N^{2D} - J^{2D}} \\
R_{N^{3D} - J^{3D}} &= R_{3D}^{N^{3D} - J^{3D}} \\
R_J^{2D} &= R_{2D}^{2D} + \sum^* (R_{3D}^{3D}) \\
R_J^{3D} &= C
\end{align*}
\]
Coupling: New Jacobian

Derivatives for new residuals (12) at the 2D interface nodes are given by (14)

\[
\frac{\partial r^\mathcal{K}}{\partial s} = \frac{\partial r_{2D}^\mathcal{K}}{\partial s} + \sum_{j \in C(\mathcal{K}) \subset J^3D} \frac{\partial r_{3D}^j}{\partial s} \quad \forall \mathcal{K} \in J^{2D}
\]  

(14)

Eq. (14) can be reinterpreted as (15) for programming purposes

\[
\text{Block row (} \mathcal{K} \text{) (New Jacobian)} = \text{Block row (} \mathcal{K} \text{) (Old Jacobian)} + \sum_{j \in C(\mathcal{K})} \text{Block row (} j \text{) (Old Jacobian)} \quad \forall \mathcal{K} \in J^{2D}
\]  

(15)
Coupling: New Jacobian

Derivatives for new residuals (12) at the 3D interface nodes are given by (16)

\[
\frac{\partial r^i}{\partial s^k} = \frac{\partial s^i_{3D}}{\partial s^i} - \frac{\partial s^i_{2D}}{\partial s^i} = \begin{cases} 
+ [I], & \text{if } i = j \\
- [I], & \text{if } i = K \\
[0], & \text{otherwise}
\end{cases} 
\forall j \in J^{3D}, \text{where} \\
\exists ! K \in J^{2D}: j \in C(K)
\] (16)

Eq. (16) can be reinterpreted as (17) for programming purposes

\[
\begin{align*}
\text{New Jacobian Block}[j][j] &= + [I] \\
\text{New Jacobian Block}[j][K] &= - [I] \\
\text{New Jacobian Block}[j][i] &= [0] \\
\end{align*}
\forall j \in J^{3D}, \text{where} \\
\exists ! K \in J^{2D}: j \in C(K) \\
\text{and } i \neq j, K
\] (17)
Coupling: New system

Use the new residuals (12), and derivatives (14) and (16), to modify the non-coupled system (11), to get the coupled system (18)

\[
\begin{bmatrix}
\frac{\partial R_{2D}^{N^2D-j^2D}}{\partial s_{2D}^{N^2D-j^2D}} & 0 & \frac{\partial R_{2D}^{N^2D-j^2D}}{\partial s_{j^2D}} \\
0 & \frac{\partial R_{3D}^{N^3D-j^3D}}{\partial s_{j^3D}} & \frac{\partial R_{3D}^{N^3D-j^3D}}{\partial s_{3D}} \\
\frac{\partial \Sigma^* (R_{3D}^{j^3D})}{\partial s_{2D}^{N^2D-j^2D}} & \frac{\partial \Sigma^* (R_{3D}^{j^3D})}{\partial s_{2D}^{j^2D}} & \frac{\partial \Sigma^* (R_{3D}^{j^3D})}{\partial s_{3D}^{j^3D}}
\end{bmatrix}
\begin{bmatrix}
\Delta s_{2D}^{N^2D-j^2D} \\
\Delta s_{3D}^{N^3D-j^3D} \\
\Delta s_{2D}^{j^2D}, \Delta s_{3D}^{j^3D}
\end{bmatrix} =
\begin{bmatrix}
R_{2D}^{N^2D-j^2D}(s_{2D}) \\
R_{3D}^{N^3D-j^3D}(s_{3D}) \\
C(s_{2D}^{j^2D}, s_{3D}^{j^3D})
\end{bmatrix}
\]

Solve (20) and update the solution vector

Check if nonlinear equations (8) are satisfied within user-defined tolerance

- If YES, then increment time step
- If NO, perform the next Newton-Raphson iteration
Example

Primary set definitions:
\[ N^{2D} = \{ \text{All nodes in 2D model} \} \]
\[ N^{3D} = \{ \text{All nodes in 3D model} \} \]

Separate Interface Nodes:
\[ I^{2D} = \{ 1_{2D}, 2_{2D}, 3_{2D} \} \]
\[ I^{3D} = \{ 1_{3D}, 2_{3D}, 3_{3D}, 4, \ldots, 9 \} \]

Node Columns:
\[ C(K = 1_{2D}) = \{ 1_{3D}, 2_{3D}, 3_{3D} \} \]
\[ C(K = 2_{2D}) = \{ 4, 5, 6 \} \]
\[ C(K = 3_{2D}) = \{ 7, 8, 9 \} \]
Coupling: Old system

Build the ‘old’ system of equations (19), as defined in (11).

\[
\begin{bmatrix}
\frac{\partial R_{2D}^{N^{2D} - J^{2D}}}{\partial s_{2D}^{N^{2D} - J^{2D}}} & 0 & \frac{\partial R_{2D}^{N^{2D} - J^{2D}}}{\partial s_{2D}^{J^{2D}}} \\
0 & \frac{\partial R_{3D}^{N^{3D} - J^{3D}}}{\partial s_{3D}^{N^{3D} - J^{3D}}} & 0 \\
\frac{\partial r_{2D}^1}{\partial s_{2D}^{N^{2D} - J^{2D}}} & \frac{\partial r_{2D}^1}{\partial s_{2D}^{J^{2D}}} & \frac{\partial r_{2D}^1}{\partial s_{2D}^{J^{2D}}} \\
\frac{\partial s_{2D}^1}{\partial s_{2D}^{N^{2D} - J^{2D}}} & \frac{\partial s_{2D}^1}{\partial s_{2D}^{J^{2D}}} & \frac{\partial s_{2D}^1}{\partial s_{2D}^{J^{2D}}} \\
\frac{\partial s_{2D}^2}{\partial s_{2D}^{N^{2D} - J^{2D}}} & \frac{\partial s_{2D}^2}{\partial s_{2D}^{J^{2D}}} & \frac{\partial s_{2D}^2}{\partial s_{2D}^{J^{2D}}} \\
\frac{\partial s_{2D}^3}{\partial s_{2D}^{N^{2D} - J^{2D}}} & \frac{\partial s_{2D}^3}{\partial s_{2D}^{J^{2D}}} & \frac{\partial s_{2D}^3}{\partial s_{2D}^{J^{2D}}} \\
[0] & \frac{\partial r_{3D}^1}{\partial s_{3D}^{N^{3D} - J^{3D}}} & \frac{\partial r_{3D}^1}{\partial s_{3D}^{J^{3D}}} \\
\vdots & \vdots & \vdots \\
0 & \frac{\partial r_{3D}^9}{\partial s_{3D}^{N^{3D} - J^{3D}}} & \frac{\partial r_{3D}^9}{\partial s_{3D}^{J^{3D}}} \\
\end{bmatrix} \begin{bmatrix}
\Delta s_{2D}^{N^{2D} - J^{2D}} \\
\Delta s_{3D}^{N^{3D} - J^{3D}} \\
\Delta r_{2D}^{1} \\
\Delta r_{2D}^{2} \\
\Delta r_{2D}^{3} \\
\Delta s_{3D}^{2} \\
\Delta s_{3D}^{3} \\
\Delta r_{3D}^{1} \\
\vdots \\
\Delta r_{3D}^{9}
\end{bmatrix} = \begin{bmatrix}
R_{2D}^{N^{2D} - J^{2D}}(s_{2D}) \\
R_{3D}^{N^{3D} - J^{3D}}(s_{3D}) \\
\{r_{2D}^{1}(s_{2D})\} \\
\{r_{2D}^{2}(s_{2D})\} \\
\{r_{2D}^{3}(s_{2D})\} \\
\{r_{3D}^{1}(s_{3D})\} \\
\vdots \\
\{r_{3D}^{9}(s_{3D})\}
\end{bmatrix}
\]

(19)
Coupling: New residuals

Build the new residuals \((20)\) using the old residuals and constraints, as per \((12)\) and \((13)\)

\[
R^{N2D-32D} = R^{N2D-32D}_2 \\
R^{N3D-33D} = R^{N3D-33D}_3
\]

\[
R^{32D} = \begin{cases} 
   r^1 = r_{2D}^1 + \sum_{i \in C(1)} r_{3D}^i = r_{2D}^1 + r_{3D}^1 + r_{3D}^2 + r_{3D}^3 \\
   r^2 = r_{2D}^2 + \sum_{i \in C(2)} r_{3D}^i = r_{2D}^2 + r_{3D}^5 + r_{3D}^6 \\
   r^3 = r_{2D}^3 + \sum_{i \in C(3)} r_{3D}^i = r_{2D}^3 + r_{3D}^7 + r_{3D}^8 + r_{3D}^9 
\end{cases}
\]

\[
R^{33D} = C = \begin{cases} 
   c^1 = s_{3D}^1 - s_{2D}^1 \\
   c^2 = s_{3D}^2 - s_{2D}^2 \\
   c^3 = s_{3D}^3 - s_{2D}^1 \\
   c^4 = s_{3D}^4 - s_{2D}^2 \\
   c^5 = s_{3D}^5 - s_{2D}^2 \\
   c^6 = s_{3D}^6 - s_{2D}^2 \\
   c^7 = s_{3D}^7 - s_{2D}^3 \\
   c^8 = s_{3D}^8 - s_{2D}^3 \\
   c^9 = s_{3D}^9 - s_{2D}^3 
\end{cases}
\]
Coupling: New Jacobian

Build the modified Jacobian (21) for the Newton-Raphson iterations (9) using the modified residuals (20) and the derivatives (14) and (16)

\[
\text{(New Jacobian)} = \frac{\partial R}{\partial s} = \begin{bmatrix}
\frac{\partial R_{2D}^{N2D_{2D}-j2D}}{\partial s_{2D}^{N2D_{2D}-j2D}} & 0 & \frac{\partial R_{3D}^{N3D_{2D}-j3D}}{\partial s_{3D}^{N3D_{2D}-j3D}} \\
0 & \frac{\partial R_{2D}^{N2D_{3D}-j2D}}{\partial s_{2D}^{N3D_{3D}-j2D}} & 0 \\
\frac{\partial r^1}{\partial s_{2D}^{N2D_{2D}-j2D}} & \frac{\partial r^1}{\partial s_{3D}^{N3D_{2D}-j3D}} & \frac{\partial r^1}{\partial s_{3D}^{N3D_{3D}-j3D}} \\
\frac{\partial r^2}{\partial s_{2D}^{N2D_{2D}-j2D}} & \frac{\partial r^2}{\partial s_{3D}^{N3D_{2D}-j3D}} & \frac{\partial r^2}{\partial s_{3D}^{N3D_{3D}-j3D}} \\
\frac{\partial r^3}{\partial s_{2D}^{N2D_{2D}-j2D}} & \frac{\partial r^3}{\partial s_{3D}^{N3D_{2D}-j3D}} & \frac{\partial r^3}{\partial s_{3D}^{N3D_{3D}-j3D}} \\
\frac{\partial c^1}{\partial s_{2D}^{N2D_{2D}-j2D}} & \frac{\partial c^1}{\partial s_{3D}^{N3D_{2D}-j3D}} & \frac{\partial c^1}{\partial s_{3D}^{N3D_{3D}-j3D}} \\
\vdots & \vdots & \vdots \\
\frac{\partial c^9}{\partial s_{2D}^{N2D_{2D}-j2D}} & \frac{\partial c^9}{\partial s_{3D}^{N3D_{2D}-j3D}} & \frac{\partial c^9}{\partial s_{3D}^{N3D_{3D}-j3D}} \\
\frac{\partial c^1}{\partial s_{2D}^{N3D_{2D}-j3D}} & \frac{\partial c^1}{\partial s_{3D}^{N3D_{2D}-j3D}} & \frac{\partial c^1}{\partial s_{3D}^{N3D_{3D}-j3D}} \\
\vdots & \vdots & \vdots \\
\frac{\partial c^9}{\partial s_{2D}^{N3D_{2D}-j3D}} & \frac{\partial c^9}{\partial s_{3D}^{N3D_{2D}-j3D}} & \frac{\partial c^9}{\partial s_{3D}^{N3D_{3D}-j3D}} \\
\frac{\partial c^1}{\partial s_{2D}^{N3D_{3D}-j3D}} & \frac{\partial c^1}{\partial s_{3D}^{N3D_{3D}-j3D}} & \frac{\partial c^1}{\partial s_{3D}^{N3D_{3D}-j3D}} \\
\vdots & \vdots & \vdots \\
\frac{\partial c^9}{\partial s_{2D}^{N3D_{3D}-j3D}} & \frac{\partial c^9}{\partial s_{3D}^{N3D_{3D}-j3D}} & \frac{\partial c^9}{\partial s_{3D}^{N3D_{3D}-j3D}}
\end{bmatrix}
\]
Coupling: New Jacobian

Expanding terms in the new Jacobian (21), making use of (15) and (17), we get (22)
Coupling: New system

The Newton-Raphson iterations (9) for the coupled system are given by (23), where we have used the coupled Jacobian (22) and coupled residuals (20):

\[
\frac{\partial R_{2D}^{N^2D-J^2D}}{\partial s_{2D}^{N^2D-J^2D}} \quad [0] \quad \frac{\partial R_{2D}^{N^2D-J^2D}}{\partial s_{2D}^{N^2D-J^2D}} \quad [0] \quad \frac{\partial R_{3D}^{N^2D-J^2D}}{\partial s_{3D}^{N^2D-J^2D}} \quad [0] \quad \frac{\partial R_{3D}^{N^2D-J^2D}}{\partial s_{3D}^{N^2D-J^2D}}
\]

\[
\frac{\delta r_{2D}^{N^2D-J^2D}}{\delta s_{2D}^{N^2D-J^2D}} + \frac{\delta r_{3D}^{N^2D-J^2D}}{\delta s_{3D}^{N^2D-J^2D}} + \frac{\delta r_{3D}^{N^2D-J^2D}}{\delta s_{3D}^{N^2D-J^2D}} = \begin{bmatrix}
\delta s_{2D}^{N^2D-J^2D} \\
\delta s_{3D}^{N^2D-J^2D}
\end{bmatrix} = -
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\delta s_{2D}^{N^2D-J^2D} \\
\delta s_{3D}^{N^2D-J^2D}
\end{bmatrix} = -
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\begin{array}{c}
\delta s_{2D}^{N^2D-J^2D} \\
\delta s_{3D}^{N^2D-J^2D}
\end{array}
\end{bmatrix} = -
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

(23)
Discussion

Conservation of quantities at the 2D-3D interface
- Enforced continuity of water surface elevation
- Enforced continuity of depth averaged velocity

Solvability of the coupled 2D-3D system
- Started with the individual (non-coupled) models, solvable upon application of boundary conditions at the interface
- New 2D interface residuals obtained by summing up linearly independent interface residuals
- New 3D interface residuals set to be linearly independent constraints

Bathymetry fixed in time
- Not applicable for sediment transport, for example

Transport of constituents in the coupled 2D-3D model
- Nearly identical treatment, 1 equation per node, per constituent, instead of 3 equations per node for SW

Continuity equation and different solution variables need separate treatment
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References


Questions?
Thank You!